fo/u fopkjr Hkh# tu] ugh vkjEHk dke] foifr n[k NkM rjr e/;e eu dj ';keA
i#"k flg ldYi dj] lgr foifr vud] ^cuk^ u NkM /;; dk] j?kcj jk[k VdAA

jfpr% ekuo /ke Z i.ksrk
lnx# Jh j.kNkMnklth egkjkt

STUDY PACKAGE

Subject : Mathematics

Topic : Sequence & Progression

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Sequence & Progression

Sequence : A sequence is a function whose domain is the set N of natural numbers. Since the domain for every sequence is the set N of natural numbers, therefore a sequence is represented by its range. If $f : N \to R$, then $f(n) = t_n$ $n \in N$ is called a sequence and is denoted by
$\{f(1), f(2), f(3), \ldots\} =$
$\{t_1, t_2, t_3, \ldots\} = \{t_n\}$

Real Sequence : A sequence whose range is a subset of R is called a real sequence.

Examples : (i) 2, 5, 8, 11, ... (ii) 4, 1, – 2, – 5, ...
(iii) 3, –9, 27, – 81, ...

Types of Sequence : On the basis of the number of terms there are two types of sequence.
(i) Finite sequences : A sequence is said to be finite if it has finite number of terms.
(ii) Infinite sequences : A sequenceis said to be infinite if it has infinite number of terms.

Solved Example # 1 Write down the sequence whose n^{th} term is

(i) $\dfrac{2^n}{n}$ (ii) $\dfrac{3+(-1)^n}{3^n}$

Solution. (i) Let $t_n = \dfrac{2^n}{n}$
put n = 1, 2, 3, 4, ... we get
$t_1 = 2, t_2 = 2, t_3 = \dfrac{8}{3}, t_4 = 4$
so the sequence is $2, 2, \dfrac{8}{3}, 4, \ldots\ldots$

(ii) Let $t_n = \dfrac{3+(-1)^n}{3^n}$
put n = 1, 2, 3, 4, ...
so the sequence is $\dfrac{2}{3}, \dfrac{4}{9}, \dfrac{2}{27}, \dfrac{4}{81}, \ldots\ldots\ldots\ldots$

Series By adding or substracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, \ldots\ldots a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + \ldots\ldots + a_n$ is a series.
Example. (i) 1 + 2 + 3 + 4 + ... + n
(ii) 2 + 4 + 8 + 16 + ...

Progression : It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the n^{th} term. Those sequences whose terms follow certain patterns are called progressions.

An arithmetic progression (A.P.):
A.P. is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then A.P. can be written as a, a + d, a + 2 d,....... a + (n – 1) d,........
Example – 4, – 1, 2, 5 ....

(i) n^{th} term of an A.P.
Let a be the first term and d be the common difference of an A.P., then
$t_n = a + (n - 1)\,d$ where $d = a_n - a_{n-1}$

Solved Example # 2 If t_{54} of an A.P. is – 61 and t_4 = 64, find t_{10}.
Solution. Let a be the first term and d be the common difference
so $t_{54} = a + 53d = -61$ ....(i)
and $t_4 = a + 3d = 64$ ....(ii)
equation (i) – (ii)
$\Rightarrow$ $50d = -125$
$d = -\dfrac{5}{2}$ $\Rightarrow$ $a = \dfrac{143}{2}$
so $t_{10} = \dfrac{143}{2} + 9\left(-\dfrac{5}{2}\right) = 49$

Solved Example # 3 Find the number of terms in the sequence 4, 12, 20, ...108.
Solution. a = 4, d = 8 so 108 = 4 + (n – 1)8 $\Rightarrow$ n = 14

(ii) The sum of first n terms of are A.P.
If a is first term and d is common difference then
$$S_n = \frac{n}{2}\,[2a + (n-1)\,d]$$
$$= \frac{n}{2}\,[a + \ell\,] = n\,t_{\left(\frac{n+1}{2}\right)},$$
where ℓ is the last term and $t_{\left(\frac{n+1}{2}\right)}$ is the middle term.

(iii) r^{th} term of an A.P. when sum of first r terms is given is $t_r = s_r - S_{r-1}$.

Solved Example # 4
Find the sum of all natural numbers divisible by 5, but less than 100.
Solution. All those numbers are 5, 10, 15, 20, ... 95.
Here a = 5 n = 19 ℓ = 95 so $S = \dfrac{19}{2}\,(5 + 95) = 950$.

Solved Example # 5

Find the sum of all the three digit natural numbers which on division by 7 leaves remainder 3.

Solution. All these numbers are 101, 108, 115, ... 997, to find n.

$997 = 101 + (n - 1)\,7 \quad \Rightarrow \quad n = 129$

so $S = \dfrac{129}{2}\,[101 + 997] = 70821.$

Solved Example # 6 The sum of n terms of two A.Ps. are in ratio $\dfrac{7n+1}{4n+27}$. Find the ratio of their 11^{th} terms.

Sol. Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of two A.P.s respectively then

$$\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_2]} = \frac{7n+1}{4n+27} \quad \Rightarrow \quad \frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_2+\left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

For ratio of 11^{th} terms

$$\frac{n-1}{2} = 10 \quad \Rightarrow \quad n = 21$$

so ratio of 11^{th} terms is $\dfrac{7(21)+1}{4(21)+27}$

$= \dfrac{148}{111}$

Solved Example # 7 If sum of n terms of a sequence is given by $S_n = 2n^2 + 3n$, find its 50^{th} term.

Solution. Let t_n is n^{th} term of the sequence so $t_n = s_n - s_{n-1}$.

$= 2n^2 + 3n - 2(n-1)^2 - 3(n-1)$

$= 4n + 1$

so $t_{50} = 201.$

Self Practice Problems :

1. Which term of the sequence 2005, 2000, 1995, 1990, 1985, ... contains the first negative term

Ans. 403.

2. For an A.P. show that

$t_m + t_{2n+m} = 2\,t_{m+n}$

3. Find the maximum sum of the A.P. 40, 38, 36, 34, 32, ... **Ans.** 420

Properties of A.P.

(i) The common difference can be zero, positive or negative.

(ii) If a, b, c are in A.P. $\Rightarrow 2b = a + c$ & if a, b, c, d are in A.P. $\Rightarrow a + d = b + c$.

(iii) Three numbers in A.P. can be taken as $a - d, a, a + d$; four numbers in A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$; five numbers in A.P. are $a - 2d, a - d, a, a + d, a + 2d$ & six terms in A.P. are $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(iv) The sum of the terms of an A.P. equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) Any term of an A.P. (except the first) is equal to half the sum of terms which are equidistant from it. $a_n = 1/2\,(a_{n-k} + a_{n+k})$, $k < n$. For $k = 1$, $a_n = (1/2)\,(a_{n-1} + a_{n+1})$;
For $k = 2$, $a_n = (1/2)\,(a_{n-2} + a_{n+2})$ and so on.

(vi) If each term of an A.P. is increased, decreased, multiplied or divided by the sA.M.e non zero number, then the resulting sequence is also an A.P..

Solved Example # 8 The sum of three numbers in A.P. is 27 and the sum of their squares is 293, find them

Solution. Let the numbers be

$a - d,\ a,\ a + d$

so $3a = 27 \quad \Rightarrow \quad a = 9$

Also $(a - d)^2 + a^2 + (a + d)^2 = 293.$

$3a^2 + 2d^2 = 293$

$d^2 = 25 \quad \Rightarrow \quad d = \pm 5$

therefore numbers are 4, 9, 14.

Solved Example # 9 If a_1, a_2, a_3, a_4, a_5 are in A.P. with common difference $\neq 0$, then find the value of $\sum_{i=1}^{5} a_i$ when $a_3 = 2$.

Solution. As a_1, a_2, a_3, a_4, a_5, are in A.P., we have

$a_1 + a_5 = a_2 + a_4 = 2a_3$.

Hence $\sum_{i=1}^{5} a_i = 10.$

Solved Example # 10 If $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A.P. prove that a^2, b^2, c^2 are also in A.P.

Solution. $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A.P.

$$\Rightarrow \quad \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a} \Rightarrow \quad \frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$$

$$\Rightarrow \quad \frac{b-a}{b+c} = \frac{c-b}{a+b} \quad \Rightarrow \quad b^2 - a^2 = c^2 - b^2 \quad \Rightarrow \quad a^2, b^2, c^2 \text{ are in A.P.}$$

Solved Example # 11 If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

Solution. Given $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.

Add 2 to each term

$\Rightarrow$ $\frac{b+c+a}{a}, \frac{c+a+b}{b}, \frac{a+b+c}{c}$ are in A.P.

divide each by a + b + c $\Rightarrow$ $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

Arithmetic Mean (Mean or Average) (A.M.):

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c.

(a) **n – Arithmetic Means Between Two Numbers:**

If a, b are any two given numbers & a, $A_1, A_2, \ldots, A_n$, b are in A.P. then $A_1, A_2, \ldots A_n$ are the n A.M.'s between a & b.

$$A_1 = a + \frac{b-a}{n+1},\ A_2 = a + \frac{2(b-a)}{n+1}, \ldots\ldots, A_n = a + \frac{n(b-a)}{n+1}$$

NOTE : Sum of n A.M.'s inserted between a & b is equal to n times the single A.M. between a & b

i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a & b.

Solved Example # 12 Between two numbers whose sum is $\frac{13}{6}$, an even number of A.M.s is inserted, the sum of these means exceeds their number by unity. Find the number of means.

Solution. Let a and b be two numbers and 2n A.M.s are inserted between a and b then

$\frac{2n}{2}(a + b) = 2n + 1.$

$n\left(\frac{13}{6}\right) = 2n + 1.$ $\left[\text{given } a + b = \frac{13}{6}\right]$

$\Rightarrow$ n = 6. $\therefore$ Number of means = 12.

Solved Example # 13 Insert 20 A.M. between 2 and 86.

Solution. Here 2 is the first term and 86 is the 22^{nd} term of A.P. so 86 = 2 + (21)d

$\Rightarrow$ d = 4

so the series is

2, 6, 10, 14,......., 82, 86 $\therefore$ required means are 6, 10, 14,...82.

Self Practice Problems :

4. If A.M. between p^{th} and q^{th} terms of an A.P. be equal to the A.M. between r^{th} and s^{th} term of the A.P. then prove that p + q = r + s.

5. If n A.M.s are inserted between 20 and 80 such that first means : last mean = 1 : 3, find n. **Ans. n = 11**

6. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$, $a \neq b$ is the A.M. of a and b. **Ans.** n = 0

Geometric Progression (G.P.)

G.P. is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the proceeding terms multiplied by a constant. Thus in a G.P. the ratio of successive terms is constant. This constant factor is called the **common ratio** of the series & is obtained by dividing any term by that which immediately proceeds it. Therefore a, ar, ar^2, ar^3, ar^4,...... is a G.P. with a as the first term & r as common ratio.

Example 2, 4, 8, 16 ....

Example $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$ ....

(i) n^{th} term = $a\,r^{n-1}$

(ii) Sum of the first n terms i.e. $S_n = \begin{cases} \frac{a(r^n-1)}{r-1}, & r \neq 1 \\ na, & r = 1 \end{cases}$

(iii) Sum of an infinite G.P. when $|r| < 1$. When $n \to \infty$ $r^n \to 0$ if $|r| < 1$ therefore, $S_\infty = \frac{a}{1-r}\ (|r| < 1)$.

Solved Example # 14 If the first term of G.P. is 7, its n^{th} term is 448 and sum of first n terms is 889, then find the fifth term of G.P.

Solution. Given a = 7 the first term

$t_n = ar^{n-1} = 7(r)^{n-1} = 448.$

$\Rightarrow$ $7r^n = 448\,r$

Also $S_n = \frac{a(r^n-1)}{r-1} = \frac{7(r^n-1)}{r-1}$ $\Rightarrow$ $889 = \frac{448r-7}{r-1}$

$\Rightarrow$ r = 2

Hence $T_5 = ar^4 = 7(2)^4 = 112.$

Solved Example # 15: The first term of an infinite G.P. is 1 and any term is equal to the sum of all the succeeding terms. Find the series.

Solution. Let the G.P. be 1, r, r^2, r^3, ...

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given condition $\Rightarrow r = \dfrac{r^2}{1-r}$ $\Rightarrow$ $r = \dfrac{1}{2}$,

Hence series is $1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \ldots\ldots\ldots\ldots\infty$

Solved Example # 16: Let $S = 1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots\ldots\ldots$ find the sum of

(i) first 20 terms of the series (ii) infinite terms of the series.

Solution. (i) $S_{20} = \dfrac{\left(1-\left(\dfrac{1}{2}\right)^{20}\right)}{1-\dfrac{1}{2}} = \dfrac{2^{20}-1}{2^{19}}$. (ii) $S_\infty = \dfrac{1}{1-\dfrac{1}{2}} = 2$.

Self Practice Problems :

1. Find the G.P. if the common ratio of G.P. is 3, n^{th} term is 486 and sum of first n terms is 728.
Ans. 2, 6, 18, 54, 162, 486.

2. If the p^{th}, q^{th}, r^{th} terms of a G.P. be a, b, c respectively, prove that $a^{q-r}\, b^{r-p}\, c^{p-q} = 1$.

3. A G.P. consist of 2n terms. If the sum of the terms occupying the odd places is S_1 and that of the terms occupying the even places is S_2 then find the common ratio of the progression. **Ans.** $\dfrac{S_2}{S_1}$.

4. The sum of infinite number of terms of a G.P. is 4, and the sum of their cubes is 192, find the series.

Ans. $6, -3, \dfrac{3}{2}, \ldots\ldots\ldots$

Properties of G.P.

(i) If a, b, c are in G.P. $\Rightarrow b^2 = ac$, in general if $a_1, a_2, a_3, a_4, \ldots\ldots\ldots a_{n-1}, a_n$ are in G.P., then $a_1a_n = a_1a_{n-1} = a_3\, a_{n-2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(ii) Any three consecutive terms of a G.P. can be taken as $\dfrac{a}{r}$, a , ar, in general we take $\dfrac{a}{r^k}, \dfrac{a}{r^{k-1}}, \dfrac{a}{r^{k-2}}, \ldots\ldots\ldots a, ar, ar^2, \ldots\ldots\ldots ar^k$ in case we have to take 2k + 1 terms in a G.P.

(iii) Any four consecutive terms of a G.P. can be taken as $\dfrac{a}{r^3}, \dfrac{a}{r}$, ar, ar^3, in general we take $\dfrac{a}{r^{2k-1}}, \dfrac{a}{r^{2k-3}}, \ldots\ldots\ldots \dfrac{a}{r}$, ar, $\ldots\ldots\ldots\ldots ar^{2k-1}$ in case we have to take 2k terms in a G.P.

(iv) If each term of a G.P. be multiplied or divided or raised to power by the some non–zero quantity, the resulting sequence is also a G.P..

(v) If $a_1, a_2, a_3, \ldots\ldots\ldots$ and $b_1, b_2, b_3, \ldots\ldots\ldots$ are two G.P's with common ratio r_1 and r_2 respectively then the sequence $a_1b_1, a_2b_2, a_3b_3, \ldots\ldots$ is also a G.P. with common ratio $r_1 r_2$.

(vi) If $a_1, a_2, a_3, \ldots\ldots\ldots$ are in G.P. where each $a_i > 0$, then $\log a_1, \log a_2, \log a_3, \ldots\ldots\ldots$ are in A.P. and its converse is also true.

Solved Example # 17: Find three numbers in G.P. having sum 19 and product 216.

Solution. Let the three numbers be $\dfrac{a}{r}$, a, ar so $a\left[\dfrac{1}{r}+1+r\right] = 19$ ....(i)

and $a^3 = 216$ $\Rightarrow$ $a = 6$

so from (i) $6r^2 - 13r + 6 = 0$.

$\Rightarrow$ $r = \dfrac{3}{2}, \dfrac{2}{3}$ Hence the three numbers are 4, 6, 9.

Solved Example # 18: Find the product of 11 terms in G.P. whose 6^{th} is 5.

Solution.: Using the property

$a_1a_{11} = a_2a_{10} = a_3a_9 = \ldots\ldots\ldots\ldots = a_6^{\,2} = 25$

Hence product of terms = 5^{11}

Solved Example # 19: Using G.P. express $0.\bar{3}$ and $1.2\bar{3}$ as $\dfrac{p}{q}$ form.

Solution. Let $x = 0.\bar{3} = 0.3333\ldots\ldots\ldots\ldots$

$= 0.3 + 0.03 + 0.003 + 0.0003 + \ldots\ldots\ldots\ldots$

$= \dfrac{3}{10} + \dfrac{3}{100} + \dfrac{3}{1000} + \dfrac{3}{10000} + \ldots\ldots\ldots\ldots$

$= \dfrac{\dfrac{3}{10}}{1-\dfrac{1}{10}} = \dfrac{3}{9} = \dfrac{1}{3}$.

Let $y = 1.2\bar{3}$

$= 1.233333$

$= 1.2 + 0.03 + 0.003 + 0.0003 + \ldots\ldots\ldots\ldots$

$= 1.2 + \dfrac{3}{10^2} + \dfrac{3}{10^3} + \dfrac{3}{10^4} + \ldots\ldots\ldots\ldots$

$= 1.2 + \dfrac{\frac{3}{10^2}}{1-\frac{1}{10}} = 1.2 + \dfrac{1}{30} = \dfrac{37}{30}$.

Solved Example # 20
Evaluate 7 + 77 + 777 + ... upto n terms.

Solution. Let S = 7 + 77 + 777 + ...upto n terms.

$= \dfrac{7}{9}$ [9 + 99 + 999 + ...]

$= \dfrac{7}{9}$ [(10 – 1) + (10² – 1) + (10³ – 1) + ... + upto n terms]

$= \dfrac{7}{9}$ [10 + 10² + 10³ + ...+ 10ⁿ – n]

$= \dfrac{7}{9}\left[10\,\dfrac{(10^n)-1}{9} - n\right] \qquad = \dfrac{7}{81}\,[10^{n+1} - 9n - 10]$

Geometric Means (Mean Proportional) (G.M.):

If a, b, c are in G.P., b is the G.M. between a & c.

$b^2 = ac$, therefore $b = \sqrt{ac}$; a > 0, c > 0.

(a) **n–Geometric Means Between a, b:**
If a, b are two given numbers & a, $G_1, G_2, \ldots, G_n$, b are in G.P.. Then $G_1, G_2, G_3, \ldots, G_n$ are n G.M.s between a & b.

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$,......, $G_n = a(b/a)^{n/n+1}$

NOTE : The product of n G.M.s between a & b is equal to the nth power of the single G.M. between a & b

i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single G.M. between a & b.

Solved Example # 21 Insert 4 G.M.s between 2 and 486.

Solution. Common ratio of the series is given by $r = \left(\dfrac{b}{a}\right)^{\frac{1}{n+1}} = (243)^{1/5} = 3$

Hence four G.M.s are 6, 18, 54, 162.

Self Practice Problems :

1. The sum of three numbers in G.P. in 70, if the two extremes be multiplied each by 4 and the mean by 5, the products are in A.P. Find the numbers. **Ans.** 10, 20, 40

2. If $a = \underbrace{111.........1}_{55}$, $b = 1 + 10 + 10^2 + 10^3 + 10^4$ and $c = 1 + 10^5 + 10^{10} + \ldots + 10^{50}$, then prove that
(i) 'a' is a composite number (ii) a = bc.

Harmonic Progression (H.P.) : A sequence is said to H.P. if the reciprocals of its terms are in A.P.. If the sequence $a_1, a_2, a_3, \ldots, a_n$ is an H.P. then $1/a_1, 1/a_2, \ldots, 1/a_n$ is an A.P. & converse. Here we do not have the formula for the sum of the n terms of a H.P.. For H.P. whose first term is a and second term is b, the nth term is $t_n = \dfrac{ab}{b+(n-1)(a-b)}$. If a, b, c are in H.P. $\Rightarrow b = \dfrac{2ac}{a+c}$ or $\dfrac{a}{c} = \dfrac{a-b}{b-c}$.

NOTE : (i) If a, b, c are in A.P. $\Rightarrow \dfrac{a-b}{b-c} = \dfrac{a}{a}$ (ii) If a, b, c are in G.P. $\Rightarrow \dfrac{a-b}{b-c} = \dfrac{a}{b}$

Harmonic Mean (H.M.):

If a, b, c are in H.P., b is the H.M. between a & c, then b = 2ac/[a + c].
If $a_1, a_2, \ldots, a_n$ are 'n' non-zero numbers then H.M. H of these numbers is given by

$\dfrac{1}{H} = \dfrac{1}{n}\left[\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots\ldots + \dfrac{1}{a_n}\right]$

Solved Example # 22: If mth term of H.P. is n, while nth term is m, find its (m + n)th term.

Solution.: Given $T_m = n$ or $\dfrac{1}{a+(m-1)d} = n$; where a is the first term and d is the common difference of the corresponding A.P.

so $a + (m-1)d = \dfrac{1}{n}$ and $a + (n-1)d = \dfrac{1}{m}$ $\Rightarrow (m-n)d = \dfrac{m-n}{mn}$ or $d = \dfrac{1}{mn}$

so $a = \dfrac{1}{n} - \dfrac{(m-1)}{mn} = \dfrac{1}{mn}$

Hence $T_{(m+n)} = \dfrac{1}{a+(m+n-1)d} = \dfrac{mn}{1+m+n-1} = \dfrac{mn}{m+n}$.

Solved Example # 23: Insert 4 H.M between 2/3 and 2/13.

Solution. Let d be the common difference of corresponding A.P. so $d = \dfrac{\frac{13}{2}-\frac{3}{2}}{5} = 1$.

$\therefore \dfrac{1}{H_1} = \dfrac{3}{2} + 1 = \dfrac{5}{2}$ or $H_1 = \dfrac{2}{5}$

$\frac{1}{H_2} = \frac{3}{2} + 2 = \frac{7}{2}$ or $H_2 = \frac{2}{7}$

$\frac{1}{H_3} = \frac{3}{2} + 3 = \frac{9}{2}$ or $H_3 = \frac{2}{9}$

$\frac{1}{H_4} = \frac{3}{2} + 4 = \frac{11}{2}$ or $H_4 = \frac{2}{11}$.

Solved Example # 24: If p^{th}, q^{th}, r^{th} terms of a H.P. be a, b, c respectively, prove that
$(q - r)bc + (r - p) ac + (p - q) ab = 0$

Solution. Let x be the first term and d be the common difference of the corresponding A.P..

so $\frac{1}{a} = x + (p - 1)d$ ....(i)

$\frac{1}{b} = x + (q - 1) d$ ....(ii)

$\frac{1}{c} = x + (r - 1) d$ ....(iii)

(i) - (ii) $\Rightarrow$ $ab(p - q)d = b - a$ ....(iv)
(ii) - (iii) $\Rightarrow$ $bc (q - r)d = c - b$ ....(v)
(iii) - (i) $\Rightarrow$ $ac (r - p) d = a - c$ ....(vi)
(iv) + (v) + (vi) gives
$bc (q - r) + ac(r - p) + ab (p - q) = 0$.

Self Practice Problems : 1. If a, b, c be in H.P., show that a : a – b = a + c : a – c.
2. If the H.M. between two quantities is to their G.M.s as 12 to 13, prove that the quantities are in ratio 4 to 9.
3. If H be the harmonic mean of a and b then find the value of $\frac{H}{2a} + \frac{H}{2b} - 1$. **Ans.** 0
4. If a, b, c, d are in H.P., the show that ab + bc + cd = 3ad

Relation between means :

(i) If A, G, H are respectively A.M., G.M., H.M. between a & b both being unequal & positive then,
$G^2 = AH$ i.e. A, G, H are in G.P.

Solved Example # 25: The A.M. of two numbers exceeds the G.M. by $\frac{3}{2}$ and the G.M. exceeds the H.M. by $\frac{6}{5}$; find the numbers.

Solution. Let the numbers be a and b, now using the relation

$G^2 = A.H.$

$= \left(G + \frac{3}{2}\right)\left(G - \frac{6}{5}\right)$

$= G^2 + \frac{3}{10} G - \frac{9}{5}$ $\Rightarrow$ $G = 6$

i.e. ab = 36
also a + b = 15 Hence the two numbers are 3 and 12.

(ii) **A.M. ≥ G.M. ≥ H.M.**
Let $a_1, a_2, a_3, ...a_n$ be n positive real numbers, then we define their

A.M. = $\frac{a_1 + a_2 + a_3 + ... + a_n}{n}$, their

G.M. = $(a_1 a_2 a_3 ....a_n)^{1/n}$ and their H.M. = $\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}$ It can be shown that

A.M. ≥ G.M. ≥ H.M. and equality holds at either places iff
$a_1 = a_2 = a_3 = ...= a_n$

Solved Example # 26 If a, b, c, > 0 prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3$

Solution. Using the relation A.M. ≥ G.M. we have

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \ge \left(\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}\right)^{\frac{1}{3}} \Rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3$$

Solved Example # 27 For non-zero x, y, z prove that $(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \ge 9$

Solution. Using the relation A.M. ≥ H.M.

$$\frac{x + y + z}{3} \ge \frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$$

$$\Rightarrow (x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \ge 9$$

Sol. Ex. # 28: If $a_i > 0 \ \forall\ i \in N$ such that $\prod_{i=1}^{n} a_i = 1$, then prove that $(1 + a_1)(1 + a_2)(1 + a_3)$ ....$(1 + a_n) \geq 2^n$

Solution. Using A.M. ≥ G.M.

$1 + a_1 \geq 2\sqrt{a_1}$

$1 + a_2 \geq 2\sqrt{a_2}$

$1 + a_n \geq 2\sqrt{a_n}$ $\Rightarrow$ $(1 + a_1)(1 + a_2)$ ....$(1 + a_n) \geq 2^n (a_1a_2a_3......a_n)^{1/n}$

As $a_1\, a_2\, a_3$ ..... $a_n = 1$

Hence $(1 + a_1)(1 + a_2)$ .... $(1 + a_n) \geq 2^n$.

Solved Example # 29 If $n > 0$ prove that $2^n > 1 + n\sqrt{2^{n-1}}$

Solution. Using the relation A.M. ≥ G.M. on the numbers 1, 2, 2^2, 2^3.......... 2^{n-1} we have

$$\frac{1+2+2^2+.......+2^{n-1}}{n} > (1.2\ 2^2\ 2^3\ ....2^{n-1})^{1/n}$$

Equality does not hold as all the numbers are not equal.

$$\Rightarrow \quad \frac{2^n-1}{2-1} > n\left(2^{\frac{(n-1)n}{2}}\right)^{\frac{1}{n}} \quad \Rightarrow \quad 2^n - 1 > n\ 2^{\frac{(n-1)}{2}}$$

$$\Rightarrow \quad 2^n > 1 + n\ 2^{\frac{(n-1)}{2}}$$

Sol. Ex. # 30 Find the greatest value of xyz for positive value of x, y, z subject to the condition xy + yz + zx = 12.

Solution. Using the relation A.M. ≥ G.M.

$$\frac{xy+yz+zx}{3} \geq (x^2\,y^2\,z^2)^{1/3} \qquad 4 \geq (x\,y\,z)^{2/3} \qquad \Rightarrow \qquad xyz \leq 8$$

Solved Example # 32 If a, b, c are in H.P. and they are distinct and positive then prove that $a^n + c^n > 2b^n$

Solution. Let a^n and c^n be two numbers

then $\frac{a^n+c^n}{2} > (a^n\,c^n)^{1/2}$

$a^n + c^n > 2\,(ac)^{n/2}$ ....(i)

Also G.M. > H.M.

i.e. $\sqrt{ac} > b$ $(ac)^{n/2} > b^n$ ....(ii)

hence from (i) and (ii) $a^n + c^n > 2b^n$

Self Practice Problems :

1. If a, b, c are real and distinct then show that $a^2(1 + b^2) + b^2(1 + c^2) + c^2(1 + a^2) > 6abc$

2. Prove that $n^n > 1 . 3 . 5$ ....$(2n - 1)$

3. If a, b, c, d be four distinct positive quantities in G.P. then show that

(i) $a + d > b + c$ (ii) $\frac{1}{ab} + \frac{1}{cd} > 2\left(\frac{1}{bd} + \frac{1}{ac} - \frac{1}{ad}\right)$

4. Prove that ΔABC is an equilateral triangle iff $\tan A + \tan B + \tan C = 3\sqrt{3}$

5. If $a, b, c > 0$ prove that $[(1 + a)(1 + b)(1 + c)]^7 > 7^7\,a^4\,b^4\,c^4$

Arithmetico-Geometric Series: A series each term of which is formed by multiplying the corresponding term of an A.P. & G.P. is called the AritH.M.etico–Geometric Series. e.g. $1 + 3x + 5x^2 + 7x^3 +$..... Here 1, 3, 5,.... are in A.P. & 1, x, x^2, x^3.... are in G.P..

Sum of n terms of an Arithmetico–Geometric Series:

Let $S_n = a + (a + d)\,r + (a + 2\,d)\,r^2 +..... + [a + (n - 1)d]\,r^{n-1}$

then $S_n = \frac{a}{1-r} + \frac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{[a+(n-1)d]\,r^n}{1-r}$, $r \neq 1$.

Sum To Infinity: If $|r| < 1$ & $n \to \infty$ then $\lim_{n\to\infty} r^n = 0$ $\Rightarrow$ $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$.

Solved Example # 33 Find the sum of the series

$$1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \text{...... to n terms.}$$

Solution. Let $S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + ... + \frac{3n-2}{5^{n-1}}$ ....(i)

$\left(\frac{1}{5}\right) S = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + ... + \frac{3n-5}{5^{n-1}} + \frac{3n-2}{5^n}$ ....(ii)

(i) – (ii) ⇒

$$\frac{4}{5}\,S = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + ... + \frac{3}{5^{n-1}} - \frac{3n-2}{5^n}.$$

$$\frac{4}{5}\,S = 1 + \frac{\frac{3}{5}\left(1-\left(\frac{1}{5}\right)^{n-1}\right)}{1-\frac{1}{5}} - \frac{3n-2}{5^n}$$

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$= 1 + \frac{3}{4} - \frac{3}{4} \times \frac{1}{5^{n-1}} - \frac{3n-2}{5^n}$

$= \frac{7}{4} - \frac{12n+7}{4.5\,n}$ $\therefore$ $S = \frac{35}{16} - \frac{(12n+7)}{16\,.\,5^{n-1}}$.

Solved Example # 35: Evaluate $1 + 2x + 3x^2 + 4x^3 + ...$ upto infinity where $| x | < 1$.

Solution. Let $S = 1 + 2x + 3x^2 + 4x^3 + ...$ ....(i)

$xS = x + 2x^2 + 3x^3 + ...$ ....(ii)

(i) - (ii) $\Rightarrow$ $(1 - x) S = 1 + x + x^2 + x^3 + ...$ or $S = \frac{1}{(1-x)^2}$

Solved Example # 36 Evaluate $1 + (1 + b) r + (1 + b + b^2) r^2 + ...$ to infinite terms for $| br | < 1$.

Solution. Let $S = 1 + (1 + b)r + (1 + b + b^2) r^2 + ...$ ....(i)

$rS = r + (1 + b) r^2 + ...$ ....(ii)

(i) - (ii) $\Rightarrow$ $(1 - r)S = 1 + br + b^2r^2 + b^3r^3 + ...$

$\Rightarrow$ $S = \frac{1}{(1-br)(1-r)}$

Self Practice Problems :

1. Evaluate $1.2 + 2.2^2 + 3.2^3 + ... + 100.\ 2^{100}$ **Ans.** $99.2^{101} + 2$.

2. Evaluate $1 + 3x + 6x^2 + 10x^3 + ...$ upto infinite term where $| x | < 1$. **Ans.** $\frac{1}{(1-x)^3}$

3. Sum to n terms of the series $1 + 2\left(1+\frac{1}{n}\right) + 3\left(1+\frac{1}{n}\right)^2 + ...$ **Ans.** n^2

Important Results

(i) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$. (ii) $\sum_{r=1}^{n} k\,a_r = k \sum_{r=1}^{n} a_r$.

(iii) $\sum_{r=1}^{n} k = k + k + k.......n$ times $= nk$; where k is a constant. (iv) $\sum_{r=1}^{n} r = 1 + 2 + 3 +.....+ n = \frac{n\,(n+1)}{2}$

(v) $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 +......+ n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$ (vi) $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 +..........+ n^3 = \frac{n^2\,(n+1)^2}{4}$

(vii) $2\sum_{i<j=1}^{n} a_i\,a_j = (a_1 + a_2 + ...+ a_n)^2 - (a_1^2 + a_2^2 + ... + a_n^2)$

Solved Example # 37: Find the sum of the series to n terms whose general term is 2n + 1.

Solution. $S_n = \Sigma T_n = \Sigma(2n + 1)$

$= 2\Sigma n + \Sigma 1$

$= \frac{2(n+1)\,n}{2} + n$ $= n^2 + 2n$ or $n(n + 2)$.

Solved Example # 38: $T_k = k^2 + 2^k$ then find $\sum_{k=1}^{n} T_k$.

Solution. $\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k$

$= \frac{n\,(n+1)\,(2n+1)}{6} + \frac{2(2^n-1)}{2-1}$ $= \frac{n\,(n+1)\,(2n+1)}{6} + 2^{n+1} - 2$.

Solved Example # 39: Find the value of the expression $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$

Solution.: $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i} j = \sum_{i=1}^{n} \frac{i\,(i+1)}{2}$

$= \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right]$

$= \frac{1}{2}\left[\frac{n\,(n+1)\,(2n+1)}{6} + \frac{n\,(n+1)}{2}\right]$

$= \frac{n\,(n+1)}{12}\,[2n + 1 + 3] = \frac{n\,(n+1)\,(n+2)}{6}$.

METHOD OF DIFFERENCE

Type – 1 Let u_1, u_2, u_3 .... be a sequence, such that $u_2 - u_1, u_3 - u_2$, ... is either an A.P. or a G.P. then nth term u_n of this sequence is obtained as follows

$S = u_1 + u_2 + u_3 + ... + u_n$ ....(i)

$S = u_1 + u_2 + ... + u_{n-1} + u_n$ ....(ii)

(i) – (ii) $\Rightarrow$ $u_n = u_1 + (u_2 - u_1) + (u_3 - u_2) + ... + (u_n - u_{n-1})$

Where the series $(u_2 - u_1) + (u_3 - u_2) + ... + (u_n - u_{n-1})$ is

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either in A.P. or in G.P. then we can find u_n and hence sum of this series as $S = \sum_{r=1}^{k} u_r$

Solved Example # 40 Find the sum to n-terms 3 + 7 + 13 + 21 + ...

Solution. Let $S = 3 + 7 + 13 + 21 + \dots + T_n$ ....(i)

$S = \quad 3 + 7 + 13 + \dots + T_{n-1} + T_n$ ....(ii)

(i) – (ii) $\Rightarrow$ $T_n = 3 + 4 + 6 + 8 + \dots + (T_n - T_{n-1})$

$= 3 + \frac{n-1}{2}[8 + (n-2)2]$

$= 3 + (n-1)(n+2)$

$= n^2 + n + 1$

Hence $S = \sum (n^2 + n + 1)$

$= \sum n^2 + \sum n + \sum 1$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n \quad = \frac{n}{3}(n^2 + 3n + 5)$

Solved Example # 41 Find the sum to n-terms 1 + 4 + 10 + 22 + ...

Solution. Let $S = 1 + 4 + 10 + 22 + \dots + T_n$ ....(i)

$S = \quad 1 + 4 + 10 + \dots + T_{n-1} + T_n$ ....(ii)

(i) – (ii) $\Rightarrow$ $T_n = 1 + (3 + 6 + 12 + \dots + T_n - T_{n-1})$

$T_n = 1 + 3\left(\frac{2^{n-1} - 1}{2-1}\right)$

$T_n = 3 \cdot 2^{n-1} - 2$

So $S = \sum T_n = 3\sum 2^{n-1} - \sum 2$

$= 3 \cdot \left(\frac{2^n - 1}{2-1}\right) - 2n \quad = 3.2^n - 2n - 3$

Type – 2 If possible express r^{th} term as difference of two terms as $t_r = f(r) - f(r \pm 1)$. This can be explained with the help of examples given below.

Solved Example # 42 Find the sum to n-terms of the series 1.2 + 2.3 + 3.4 + ...

Solution. Let T_r be the general term of the series

So $T_r = r(r+1)$.

To express $t_r = f(r) - f(r-1)$ multiply and divide t_r by $[(r+2) - (r-1)]$

so $T_r = \frac{r}{3}(r+1)[(r+2) - (r-1)]$

$= \frac{1}{3}[r(r+1)(r+2) - (r-1)r(r+1)]$.

Let $f(r) = \frac{1}{3} r(r+1)(r+2)$

so $T_r = [f(r) - f(r-1)]$. Now $S = \sum_{r=1}^{n} T_r = T_1 + T_2 + T_3 + \dots + T_n$

$T_1 = \frac{1}{3}[1 \cdot 2 \cdot 3 - 0]$, $T_2 = \frac{1}{3}[2 \cdot 3 \cdot 4 - 1 \cdot 2 \cdot 3]$, $T_3 = \frac{1}{3}[3 \cdot 4 \cdot 5 - 2 \cdot 3 \cdot 4]$

$T_n = \frac{1}{3}[n(n+1)(n+2) - (n-1)n(n+1)]$ $\quad \therefore \quad S = \frac{1}{3} n(n+1)(n+2)$

Hence sum of series is f(n) – f(0).

Solved Example # 43 Sum to n terms of the series $\frac{1}{(1+x)(1+2x)} + \frac{1}{(1+2x)(1+3x)} + \frac{1}{(1+3x)(1+4x)} + \dots$

Solution. Let T_r be the general term of the series

$T_r = \frac{1}{(1+rx)(1+(r+1)x)}$ So $T_r = \frac{1}{x}\left[\frac{[1+(r+1)x] - (1+rx)}{(1+rx)(1+(r+1)x)}\right]$

$= \frac{1}{x}\left[\frac{1}{1+rx} - \frac{1}{1+(r+1)x}\right]$

$T_r = f(r) - f(r+1)$

$\therefore$ $S = \sum T_r = T_1 + T_2 + T_3 + \dots + T_n$

$= \frac{1}{x}\left[\frac{1}{1+x} - \frac{1}{1+(n+1)x}\right] \quad = \frac{n}{(1+x)[1+(n+1)x]}$

Solved Example # 44 Sun to n terms of the series $\frac{4}{1.2.3} + \frac{5}{2.3.4} + \frac{6}{3.4.5} + \dots$

Solution. Let $T_r = \frac{r+3}{r(r+1)(r+2)}$

$= \frac{1}{(r+1)(r+2)} + \frac{3}{r(r+1)(r+2)} \quad = \left[\frac{1}{r+1} - \frac{1}{r+2}\right] + \frac{3}{2}\left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right]$

$\therefore \quad S = \left[\frac{1}{2} - \frac{1}{n+2}\right] + \frac{3}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right]$

$= \frac{5}{4} - \frac{1}{n+2}\left[1+\frac{3}{2(n+1)}\right]$ $\quad = \frac{5}{4} - \frac{1}{2(n+1)(n+2)}$ [2n + 5]

Note : It is not always necessary that the series of first order of differences i.e. $u_2 - u_1$, $u_3 - u_2$ .... $u_n - u_{n-1}$, is always either in A.P. or in G.P. in such case let $u_1 = T_1$, $u_2 - u_1 = T_2$, $u_3 - u_2 = T_3$ ...., $u_n - u_{n-1} = T_n$.

So $u_n = T_1 + T_2 + ...+ T_n$ ....(i)

$u_n = \quad T_1 + T_2 + ...+ T_{n-1} + T_n$ ....(ii)

(i) – (ii) ⇒

$T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + ... + (T_n - T_{n-1})$

Now, the series $(T_2 - T_1) + (T_3 - T_2) + ... + (T_n - T_{n-1})$ is series of second order of differences and when it is either in A.P. or in G.P. , then $u_n = u_1 + \sum T_r$

Otherwise in the similar way we find series of higher order of differences and the n^{th} term of the series. With the help of following example this can be explained.

Solved Example # 45 Find the nth term and the sum of n term of the series

2, 12, 36, 80, 150, 252

Solution. Let $S = 2 + 12 + 36 + 80 + 150 + 252 + ...+T_n$ ....(i)

$S = \quad 2 + 12 + 36 + 80 + 150 + 252 + ...+T_{n-1} + T_n$ ....(ii)

(i) – (ii) ⇒ $T_n = 2 + 10 + 24 + 44 + 70 + 102 + ... + (T_n - T_{n-1})$ ....(iii)

$T_n = \quad 2 + 10 + 24 + 44 + 70 + 102 + ... + (T_{n-1} - T_{n-2}) + (T_n - T_{n-1})$ ....(iv)

(iii) – (iv) ⇒ $T_n - T_{n-1} = 2 + 8 + 14 + 20 + 26 + ...$

$= \frac{n}{2}[4 + (n-1)\,6] = n\,[3n - 1] = T_n - T_{n-1} = 3n^2 - n$

∴ general term of given series is $\sum T_n - T_{n-1} = \sum 3n^2 - n = n^3 + n^2$.

Hence sum of this series is $S = \sum n^3 + \sum n^2$

$= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} \quad = \frac{n\,(n+1)}{12}\,(3n^2 + 7n + 2)$

$\frac{1}{12}\,n\,(n+1)\,(n+2)\,(3n+1)$

Solved Example # 46: Find the general term and sum of n terms of the series 9, 16, 29, 54, 103

Sol. Let $S = 9 + 16 + 29 + 54 + 103 + ... + T_n$ ....(i)

$S = \quad 9 + 16 + 29 + 54 + 103 + ... + T_{n-1} + T_n$ ....(ii)

(i) – (ii) ⇒ $T_n = 9 + 7 + 13 + 25 + 49 + ... + (T_n - T_{n-1})$ ....(iii)

$T_n = \quad 9 + 7 + 13 + 25 + 49 + ... + (T_{n-1} - T_{n-2}) + (T_n - T_{n-1})$ ....(iv)

(iii) – (iv) ⇒ $T_n - T_{n-1} = 9 + (-2) + \underbrace{6 + 12 + 24 + ...}_{(n-2)\text{ terms}} = 7 + 6\,[2^{n-2} - 1] = 6(2)^{n-2} + 1.$

∴ General term is $T_n = 6(2)^{n-1} + n + 2$

Also sum $S = \sum T_n \quad = 6\sum 2^{n-1} + \sum n + \sum 2$

$= 6 \cdot \frac{(2^n - 1)}{2-1} + \frac{n\,(n+1)}{2} + 2n \quad = 6(2^n - 1) + \frac{n(n+5)}{2}$

Self Practice Problems : 1. Sum to n terms the following series

(i) $\frac{1}{1^3} + \frac{1+2}{1^3+2^3} + \frac{1+2+3}{1^3+2^3+3^3} + ...$ **Ans.** $\frac{2n}{n+1}$

(ii) $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + ...$ **Ans.** $\frac{1}{4}\left[\frac{1}{3} - \frac{1}{(2n+1)(2n+3)}\right]$

(iii) 1 . 5 . 9 + 2 . 6 . 10 + 3 . 7. 11 + ... **Ans.** $\frac{n}{4}\,(n+1)\,(n+8)\,(n+9)$

(iv) 4 + 14 + 30 + 52 + 82 + 114 + ... **Ans.** $n(n+1)^2$

(v) 2 + 5 + 12 + 31 + 86 + ... **Ans.** $\frac{3^n + n^2 + n - 1}{2}$

SHORT REVESION
(SEQUENCES AND SERIES)

DEFINITION : A sequence is a set of terms in a definite order with a rule for obtaining the terms.
e.g. 1 , 1/2 , 1/3 , ... , 1/n , ... is a sequence.

AN ARITHMETIC PROGRESSION (AP) : AP is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then AP can be written as a, a + d, a + 2 d, ... a + (n – 1)d, ...

n^{th} term of this AP $t_n = a + (n - 1)d$, where $d = a_n - a_{n-1}$.

The sum of the first n term of the AP is given by ; $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l]$.

where l is the last term.

NOTES :(i) If each term of an A.P. is increased, decreased, multiplied or divided by the same non zero number, then the resulting sequence is also an AP.

(ii) Three numbers in AP can be taken as a – d , a , a + d ; four numbers in AP can be taken as a – 3d, a – d, a + d, a + 3d ; five numbers in AP are a – 2d , a – d , a, a + d, a + 2d & six terms in AP are a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d etc.

(iii) The common difference can be zero, positive or negative.

(iv) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it.

(vi) $t_r = S_r - S_{r-1}$

(vii) If a , b , c are in AP $\Rightarrow$ 2 b = a + c.

GEOMETRIC PROGRESSION (GP) : GP is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the proceeding terms multiplied by a constant . Thus in a GP the ratio of successive terms is constant. This constant factor is called the **COMMON RATIO** of the series & is obtained by dividing any term by that which immediately proceeds it. Therefore a, ar, ar^2, ar^3, ar^4, ... is a GP with a as the first term & r as common ratio.

(i) n^{th} term = $a\,r^{n-1}$

(ii) Sum of the I^{st} n terms i.e. $S_n = \frac{a(r^n - 1)}{r - 1}$, if $r \neq 1$.

(iii) Sum of an infinite GP when $|r| < 1$ when $n \to \infty$ $r^n \to 0$ if $|r| < 1$ therefore, $S_\infty = \frac{a}{1-r}\ (|r| < 1)$.

(iv) If each term of a GP be multiplied or divided by the same non-zero quantity, the resulting sequence is also a GP.

(v) Any 3 consecutive terms of a GP can be taken as a/r, a, ar ; any 4 consecutive terms of a GP can be taken as a/r^3, a/r, ar, ar^3 & so on.

(vi) If a, b, c are in GP $\Rightarrow b^2 = ac$.

HARMONIC PROGRESSION (HP) : A sequence is said to HP if the reciprocals of its terms are in AP. If the sequence $a_1, a_2, a_3, ... , a_n$ is an HP then $1/a_1, 1/a_2, ... , 1/a_n$ is an AP & converse. Here we do not have the formula for the sum of the n terms of an HP. For HP whose first term is a & second term is b, the n^{th} term is $t_n = \frac{ab}{b + (n-1)(a-b)}$.

If a, b, c are in HP $\Rightarrow b = \frac{2ac}{a + c}$ or $\frac{a}{c} = \frac{a - b}{b - c}$.

MEANS

ARITHMETIC MEAN : If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c .

AM for any n positive number $a_1, a_2, ... , a_n$ is ; $A = \frac{a_1 + a_2 + a_3 + ... + a_n}{n}$.

n - ARITHMETIC MEANS BETWEEN TWO NUMBERS :
If a, b are any two given numbers & a, $A_1, A_2, ... , A_n$, b are in AP then $A_1, A_2, ... A_n$ are the n AM's between a & b .

$A_1 = a + \frac{b - a}{n + 1}$, $A_2 = a + \frac{2(b - a)}{n + 1}$, ... , $A_n = a + \frac{n(b - a)}{n + 1}$

= a + d , = a + 2 d , ... , A_n = a + nd , where $d = \frac{b-a}{n+1}$

NOTE : Sum of n AM's inserted between a & b is equal to n times the single AM between a & b

i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b.

GEOMETRIC MEANS : If a, b, c are in GP, b is the GM between a & c.

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$b^2 = ac$, therefore $b = \sqrt{ac}$; $a > 0$, $c > 0$.

n-GEOMETRIC MEANS BETWEEN a, b :

If a, b are two given numbers & a, $G_1, G_2, ..., G_n$, b are in GP. Then
$G_1, G_2, G_3, ..., G_n$ are n GMs between a & b.
$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$, ... , $G_n = a(b/a)^{n/n+1}$
$= ar$, $= ar^2$, ... $= ar^n$, where $r = (b/a)^{1/n+1}$

NOTE : The product of n GMs between a & b is equal to the n^{th} power of the single GM between a & b

i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a & b.

HARMONIC MEAN : If a, b, c are in HP, b is the HM between a & c, then $b = 2ac/[a + c]$.

THEOREM : If A, G, H are respectively AM, GM, HM between a & b both being unequal & positive then,
(i) $G^2 = AH$ **(ii)** $A > G > H$ $(G > 0)$. Note that A, G, H constitute a GP.

ARITHMETICO-GEOMETRIC SERIES :

A series each term of which is formed by multiplying the corresponding term of an AP & GP is called the **Arithmetico-Geometric Series**. e.g. $1 + 3x + 5x^2 + 7x^3 + ...$
Here 1, 3, 5, ... are in AP & 1, x, x^2, x^3 ..... are in GP.
Standart appearance of an Arithmetico-Geometric Series is
Let $S_n = a + (a + d) r + (a + 2d) r^2 + ... + [a + (n - 1)d] r^{n-1}$

SUM TO INFINITY : If $|r| < 1$ & $n \to \infty$ then $\lim_{n\to\infty} r^n = 0$. $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$.

SIGMA NOTATIONS

THEOREMS :(i) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$. **(ii)** $\sum_{r=1}^{n} k\,a_r = k \sum_{r=1}^{n} a_r$.

(iii) $\sum_{r=1}^{n} k = nk$; where k is a constant.

RESULTS

(i) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first n natural nos.)

(ii) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(iii) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(iv) $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

METHOD OF DIFFERENCE : If $T_1, T_2, T_3, ... , T_n$ are the terms of a sequence then some times the terms $T_2 - T_1, T_3 - T_2, ...$ constitute an AP/GP. n^{th} term of the series is determined & the sum to n terms of the sequence can easily be obtained.

Remember that to find the sum of n terms of a series each term of which is composed of r factors in AP, the first factors of several terms being in the same AP, we "write down the nth term, affix the next factor at the end, divide by the number of factors thus increased and by the common difference and add a constant. Determine the value of the constant by applying the initial conditions".

EXERCISE–1

Q.1 If the 10th term of an HP is 21 & 21^{st} term of the same HP is 10, then find the 210^{th} term.

Q.2 Show that $ln\,(4 \times 12 \times 36 \times 108 \times ....$ up to n terms$) = 2n\,ln\,2 + \frac{n(n-1)}{2} ln\,3$

Q.3 There are n AM's between 1 & 31 such that 7th mean : $(n-1)^{th}$ mean = 5 : 9, then find the value of n.

Q.4 Find the sum of the series , 7 + 77 + 777 + ... to n terms.

Q.5 Express the recurring decimal $0.1\overline{576}$ as a rational number using concept of infinite geometric series.

Q.6 Find the sum of the n terms of the sequence $\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + ...$

Q.7 The first term of an arithmetic progression is 1 and the sum of the first nine terms equal to 369. The first and the ninth term of a geometric progression coincide with the first and the ninth term of the arithmetic progression. Find the seventh term of the geometric progression.

Q.8 If the p^{th}, q^{th} & r^{th} terms of an AP are in GP . Show that the common ratio of the GP is $\frac{q-r}{p-q}$.

Q.9 If one AM 'a' & two GM's p & q be inserted between any two given numbers then show that $p^3 + q^3 = 2\,apq$.

Q.10 The sum of n terms of two arithmetic series are in the ratio of $(7n+1):(4n+27)$. Find the ratio of their n^{th} term.

Q.11 If S be the sum, P the product & R the sum of the reciprocals of a GP, find the value of $P^2\left(\frac{R}{S}\right)^n$.

Q.12 The first and last terms of an A.P. are a and b. There are altogether (2n + 1) terms. A new series is formed by multiplying each of the first 2n terms by the next term. Show that the sum of the new series is $\frac{(4n^2-1)(a^2+b^2)+(4n^2+2)ab}{6n}$.

Q.13 In an AP of which 'a' is the Ist term, if the sum of the Ist p terms is equal to zero, show that the sum of the next q terms is $-a(p+q)q/(p-1)$.

Q.14(a) The interior angles of a polygon are in AP. The smallest angle is 120° & the common difference is 5°. Find the number of sides of the polygon.

(b) The interior angles of a convex polygon form an arithmetic progression with a common difference of 4°. Determine the number of sides of the polygon if its largest interior angle is 172°.

Q.15 An AP & an HP have the same first term, the same last term & the same number of terms; prove that the product of the r^{th} term from the beginning in one series & the r^{th} term from the end in the other is independent of r.

Q.16 Find three numbers a, b, c between 2 & 18 such that;
(i) their sum is 25 (ii) the numbers 2, a, b are consecutive terms of an AP &
(iii) the numbers b, c, 18 are consecutive terms of a GP.

Q.17 Given that $a^x = b^y = c^z = d^u$ & a, b, c, d are in GP, show that x, y, z, u are in HP.

Q.18 In a set of four numbers, the first three are in GP & the last three are in AP, with common difference 6. If the first number is the same as the fourth, find the four numbers.

Q.19 Find the sum of the first n terms of the sequence: $1+2\left(1+\frac{1}{n}\right)+3\left(1+\frac{1}{n}\right)^2+4\left(1+\frac{1}{n}\right)^3+\ldots\ldots\ldots$

Q.20 Find the nth term and the sum to n terms of the sequence:
(i) $1+5+13+29+61+\ldots\ldots$ (ii) $6+13+22+33+\ldots\ldots.$

Q.21 The AM of two numbers exceeds their GM by 15 & HM by 27. Find the numbers.

Q.22 The harmonic mean of two numbers is 4. The airthmetic mean A & the geometric mean G satisfy the relation $2A+G^2=27$. Find the two numbers.

Q.23 Sum the following series to n terms and to infinity:

(i) $\frac{1}{1.4.7}+\frac{1}{4.7.10}+\frac{1}{7.10.13}+\ldots\ldots\ldots$ (ii) $\sum_{r=1}^{n} r(r+1)(r+2)(r+3)$

(iii) $\sum_{r=1}^{n}\frac{1}{4r^2-1}$ (iv) $\frac{1}{4}+\frac{1.3}{4.6}+\frac{1.3.5}{4.6.8}+\ldots\ldots\ldots.$

Q.24 Find the value of the sum (a) $\sum_{r=1}^{n}\sum_{s=1}^{n}\delta_{rs}\,2^r 3^s$ where δ_{rs} is zero if $r\neq s$ & δ_{rs} is one if $r=s$.

(b) $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1.$

Q.25 For or $0<\phi<\pi/2$, if:
$x=\sum_{n=0}^{\infty}\cos^{2n}\phi,\ y=\sum_{n=0}^{\infty}\sin^{2n}\phi,\ z=\sum_{n=0}^{\infty}\cos^{2n}\phi\,\sin^{2n}\phi$ then: Prove that
(i) $xyz = xy + z$ (ii) $xyz = x + y + z$

EXERCISE–2

Q.1 The series of natural numbers is divided into groups (1), (2, 3, 4), (5, 6, 7, 8, 9), ... & so on. Show that the sum of the numbers in the n^{th} group is $(n-1)^3+n^3$.

Q.2 The sum of the squares of three distinct real numbers, which are in GP is S^2. If their sum is αS, show that $\alpha^2\in(1/3,1)\cup(1,3)$.

Q.3 If there be m AP's beginning with unity whose common difference is 1, 2, 3 .... m. Show that the sum of their n^{th} terms is $(m/2)(mn-m+n+1)$.

Q.4 If S_n represents the sum to n terms of a GP whose first term & common ratio are a & r respectively, then prove that $S_1+S_3+S_5+\ldots..+S_{2n-1}=\frac{an}{1-r}-\frac{ar(1-r^{2n})}{(1-r)^2(1+r)}$.

Q.5 A geometrical & harmonic progression have the same p^{th}, q^{th} & r^{th} terms a, b, c respectively. Show that $a(b-c)\log a+b(c-a)\log b+c(a-b)\log c=0$.

Q.6 A computer solved several problems in succession. The time it took the computer to solve each successive problem was the same number of times smaller than the time it took to solve the preceding problem. How many problems were suggested to the computer if it spent 63.5 min to solve all the problems except for the first, 127 min to solve all the problems except for the last one, and 31.5 min to solve all the problems except for the first two?

Q.7 If the sum of m terms of an AP is equal to the sum of either the next n terms or the next p terms of the same AP prove that $(m+n)[(1/m)-(1/p)]=(m+p)[(1/m)-(1/n)]$ $(n\neq p)$

Q.8 If the roots of $10x^3-cx^2-54x-27=0$ are in harmonic progression, then find c & all the roots.

Q.9(a) Let a_1, a_2, a_3 ...... a_n be an AP . Prove that :

$$\frac{1}{a_1\, a_n}+\frac{1}{a_2\, a_{n-1}}+\frac{1}{a_3\, a_{n-2}}+\ldots\ldots\ldots+\frac{1}{a_n\, a_1}=\frac{2}{a_1+a_n}\left[\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots\ldots\ldots+\frac{1}{a_n}\right]$$

(b) Show that in any arithmetic progression a_1, a_2, a_3 ....

$a_1^2 - a_2^2 + a_3^2 - a_4^2 +$ ...... $+ a^2_{2K-1} - a^2_{2K} = [K/(2K-1)]\,(a_1^2 - a^2_{2K})$.

Q.10 Let a_1, a_2, ... , a_n, a_{n+1}, ... be an A.P.

Let $S_1 = a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots + a_n$

$S_2 = a_{n+1} + a_{n+2} + \ldots\ldots\ldots\ldots\ldots + a_{2n}$

$S_3 = a_{2n+1} + a_{2n+2} + \ldots\ldots\ldots\ldots + a_{3n}$

....

....

Prove that the sequence S_1, S_2, S_3, ... is an arithmetic progression whose common difference is n^2 times the common difference of the given progression.

Q.11 If a, b, c are in HP, b, c, d are in GP & c, d, e are in AP, Show that $e = ab^2/(2a-b)^2$.

Q.12 If a, b, c, d, e be 5 numbers such that a, b, c are in AP ; b, c, d are in GP & c, d, e are in HP then:

(i) Prove that a, c, e are in GP . (ii) Prove that $e = (2b-a)^2/a$.

(iii) If a = 2 & e = 18 , find all possible values of b , c , d .

Q.13 The sequence a_1, a_2, a_3, ... a_{98} satisfies the relation $a_{n+1} = a_n + 1$ for n = 1, 2, 3, ...97 and has the sum equal to 4949. Evaluate $\sum_{k=1}^{49} a_{2k}$.

Q.14 If n is a root of the equation $x^2(1-ac) - x(a^2+c^2) - (1+ac) = 0$ & if n HM's are inserted between a & c, show that the difference between the first & the last mean is equal to $ac(a-c)$.

Q.15 (a) The value of $x + y + z$ is 15 if a, x, y, z, b are in AP while the value of ; $(1/x)+(1/y)+(1/z)$ is 5/3 if a, x, y, z, b are in HP . Find a & b .

(b) The values of xyz is 15/2 or 18/5 according as the series a, x, y, z, b is an AP or HP . Find the values of a & b assuming them to be positive integer .

Q.16 An AP , a GP & a HP have 'a' & 'b' for their first two terms . Show that their $(n+2)^{th}$ terms will be in GP if $\frac{b^{2n+2}-a^{2n+2}}{ba\left(b^{2n}-a^{2n}\right)}=\frac{n+1}{n}$.

Q.17 Prove that the sum of the infinite series $\frac{1.3}{2}+\frac{3.5}{2^2}+\frac{5.7}{2^3}+\frac{7.9}{2^4}+\ldots\ldots\ldots\infty=23$.

Q.18 If there are n quantities in GP with common ratio r & S_m denotes the sum of the first m terms, show that the sum of the products of these m terms taken two & two together is $[r/(r+1)]\,[S_m]\,[S_{m-1}]$.

Q.19 Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in A.P. and hence solve the equation $x^3 - 12x^2 + 39x - 28 = 0$.

Q.20 If $ax^2 + 2bx + c = 0$ & $a_1x^2 + 2b_1x + c_1 = 0$ have a common root & a/a_1 , b/b_1 , c/c_1 are in AP, show that a_1 , b_1 & c_1 are in GP.

Q.21 If a , b , c be in GP & $\log_c a$, $\log_b c$, $\log_a b$ be in AP , then show that the common difference of the AP must be 3/2.

Q.22 If $a_1 = 1$ & for $n > 1$, $a_n = a_{n-1} + \frac{1}{a_{n-1}}$, then show that $12 < a_{75} < 15$.

Q.23 Sum to n terms :

(i) $\frac{1}{x+1}+\frac{2x}{(x+1)\,(x+2)}+\frac{3x^2}{(x+1)\,(x+2)\,(x+3)}+\ldots\ldots.$

(ii) $\frac{a_1}{1+a_1}+\frac{a_2}{(1+a_1)\,(1+a_2)}+\frac{a_3}{(1+a_1)\,(1+a_2)\,(1+a_3)}+\ldots\ldots.$

Q.24 In a GP the ratio of the sum of the first eleven terms to the sum of the last eleven terms is 1/8 and the ratio of the sum of all the terms without the first nine to the sum of all the terms without the last nine is 2. Find the number of terms in the GP.

Q.25 Given a three digit number whose digits are three successive terms of a G.P. If we subtract 792 from it, we get a number written by the same digits in the reverse order . Now if we subtract four from the hundred's digit of the initial number and leave the other digits unchanged, we get a number whose digits are successive terms of an A.P. Find the number.

EXERCISE–3

Q.1 For any odd integer $n \geq 1$, $n^3 - (n-1)^3 +$ ..... $+ (-1)^{n-1}\, 1^3 =$ ______ . [JEE '96, 1]

Q.2 $x = 1 + 3a + 6a^2 + 10a^3 +$ ..... $|a| < 1$

$y = 1 + 4b + 10b^2 + 20b^3 +$ ..... $|b| < 1$, find $S = 1 + 3ab + 5(ab)^2 +$ .... in terms of x & y.

Q.3 The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P . Find the intervals in which β and γ lie . [JEE '96, 3]

Q.4 Let p & q be roots of the equation $x^2 - 2x + A = 0$, and let r & s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmatic progression, then A = _______, and B = _______.

Q.5 a, b, c are the first three terms of a geometric series. If the harmonic mean of a & b is 12 and that of b

& c is 36, find the first five terms of the series. [REE '98, 6]

Q.6 Select the correct alternative(s). [JEE '98, 2 + 2 + 8]

(a) Let T_r be the r^{th} term of an AP, for r = 1, 2, 3, ... If for some positive integers m, n we have $T_m = \frac{1}{n}$ & $T_n = \frac{1}{m}$, then T_{mn} equals :

(A) $\frac{1}{mn}$ (B) $\frac{1}{m} + \frac{1}{n}$ (C) 1 (D) 0

(b) If x = 1, y > 1, z > 1 are in GP, then $\frac{1}{1+\ell n x}, \frac{1}{1+\ell n y}, \frac{1}{1+\ell n z}$ are in :

(A) AP (B) HP (C) GP (D) none of the above

(c) Prove that a triangle ABC is equilateral if & only if $\tan A + \tan B + \tan C = 3\sqrt{3}$.

Q.7(a) The harmonic mean of the roots of the equation $\left(5+\sqrt{2}\right) x^2 - \left(4+\sqrt{5}\right) x + 8 + 2\sqrt{5} = 0$ is

(A) 2 (B) 4 (C) 6 (D) 8

(b) Let $a_1, a_2, ..., a_{10}$, be in A.P. & $h_1, h_2, ..., h_{10}$ be in H.P. If $a_1 = h_1 = 2$ & $a_{10} = h_{10} = 3$ then $a_4 h_7$ is:

(A) 2 (B) 3 (C) 5 (D) 6

Q.8 The sum of an infinite geometric series is 162 and the sum of its first n terms is 160. If the inverse of its common ratio is an integer, find all possible values of the common ratio, n and the first terms of the series.

Q.9(a) Consider an infinite geometric series with first term 'a' and common ratio r . If the sum is 4 and the second term is 3/4, then :

(A) $a = \frac{7}{4}$, $r = \frac{3}{7}$ (B) $a = 2$, $r = \frac{3}{8}$

(C) $a = \frac{3}{2}$, $r = \frac{1}{2}$ (D) $a = 3$, $r = \frac{1}{4}$

(b) If a, b, c, d are positive real numbers such that a + b + c + d = 2, then M = (a + b) (c + d) satisfies the relation :

(A) $0 \le M \le 1$ (B) $1 \le M \le 2$

(C) $2 \le M \le 3$ (D) $3 \le M \le 4$

(c) The fourth power of the common difference of an arithmetic progression with integer entries added to the product of any four consecutive terms of it . Prove that the resulting sum is the square of an integer.

[JEE 2000, Mains, 4 out of 100]

Q.10 Given that α, γ are roots of the equation, $Ax^2 - 4x + 1 = 0$ and β, δ the roots of the equation, $Bx^2 - 6x + 1 = 0$, find values of A and B, such that α, β, γ & δ are in H.P.

[REE 2000, 5 out of 100]

Q.11 The sum of roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of squares of their reciprocals. Find whether bc^2, ca^2 and ab^2 in A.P., G.P. or H.P.? [REE 2001, 3 out of 100]

Q.12 Solve the following equations for x and y

$\log_2 x + \log_4 x + \log_{16} x + ... = y$

$\frac{5+9+13+..........+(4y+1)}{1+3+5+.............+(2y-1)} = 4\log_4 x$ [REE 2001, 5 out of 100]

Q.13(a) Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in G.P., then the integral values of p and q respectively, are

(A) –2, –32 (B) –2, 3 (C) –6, 3 (D) –6, –32

(b) If the sum of the first 2n terms of the A.P. 2, 5, 8, ... is equal to the sum of the first n terms of the A.P. 57, 59, 61, ..., then n equals

(A) 10 (B) 12 (C) 11 (D) 13

(c) Let the positive numbers a, b, c, d be in A.P. Then abc, abd, acd, bcd are

(A) NOT in A.P./G.P./H.P. (B) in A.P.

(C) in G.P. (D) H.P. [JEE 2001, Screening, 1 + 1 + 1 out of 35]

(d) Let a_1, a_2 .... be positive real numbers in G.P. For each n, let A_n, G_n, H_n, be respectively, the arithmetic mean, geometric mean and harmonic mean of a_1, a_2, a_3, ...a_n. Find an expression for the G.M. of G_1, G_2, ...G_n in terms of A_1, A_2 ....A_n, H_1, H_2, ...H_n.

Q.14(a) Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. If a < b < c and $a + b + c = \frac{3}{2}$, then the value of a is

(A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{2\sqrt{3}}$ (C) $\frac{1}{2} - \frac{1}{\sqrt{3}}$ (D) $\frac{1}{2} - \frac{1}{\sqrt{2}}$

(b) Let a, b be positive real numbers. If a , A_1 , A_2 , b are in A.P. ; a , a_1 , a_2 , b are in G.P. and a , H_1 , H_2 , b are in H.P. , show that

$\frac{G_1 G_2}{H_1 H_2} = \frac{A_1 + A_2}{H_1 + H_2} = \frac{(2a+b)(a+2b)}{9ab}$ [JEE 2002 , Mains , 5 out of 60]

Q.15 If a, b, c are in A.P., a^2, b^2, c^2 are in H.P., then prove that either a = b = c or a, b, $-\frac{c}{2}$ form a G.P.

Q.16 The first term of an infinite geometric progression is x and its sum is 5. Then
(A) $0 \le x \le 10$ (B) $0 < x < 10$ (C) $-10 < x < 0$ (D) $x > 10$

Q.17 If a, b, c are positive real numbers, then prove that $[(1 + a)(1 + b)(1 + c)]^7 > 7^7 a^4 b^4 c^4$.

Q.18(a) In the quadratic equation $ax^2 + bx + c = 0$, if $\Delta = b^2 - 4ac$ and $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$ are in G.P. where α, β are the roots of $ax^2 + bx + c = 0$, then
(A) $\Delta \ne 0$ (B) $b\Delta = 0$ (C) $c\Delta = 0$ (D) $\Delta = 0$

(b) If total number of runs scored in *n* matches is $\left(\frac{n+1}{4}\right)(2^{n+1} - n - 2)$ where n > 1, and the runs scored in the k^{th} match are given by $k \cdot 2^{n+1-k}$, where $1 \le k \le n$. Find n. [JEE 2005 (Mains), 2]

Q.19 If $A_n = \left(\frac{3}{4}\right) - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + ... + (-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $B_n = 1 - A_n$, then find the minimum natural number n_0 such that $B_n > A_n$. $\forall\, n > n_0$. [JEE 2006, 6]

EXERCISE–4

Part : (A) Only one correct option

1. If x ε R, the numbers $5^{1+x} + 5^{1-x}$, a/2, $25^x + 25^{-x}$ form an A.P. then 'a' must lie in the interval:
(A) [1, 5] (B) [2, 5] (C) [5, 12] (D) [12, ∞)

2. If x > 1 and $\left(\frac{1}{x}\right)^a$, $\left(\frac{1}{x}\right)^b$, $\left(\frac{1}{x}\right)^c$ are in G.P., then a, b, c are in
(A) A.P. (B) G.P. (C) H.P. (D) none of these

3. If A, G & H are respectively the A.M., G.M. & H.M. of three positive numbers a, b, & c, then the equation whose roots are a, b, & c is given by:
(A) $x^3 - 3Ax^2 + 3G^3x - G^3 = 0$ (B) $x^3 - 3Ax^2 + 3(G^3/H)x - G^3 = 0$
(C) $x^3 + 3Ax^2 + 3(G^3/H)x - G^3 = 0$ (D) $x^3 - 3Ax^2 - 3(G^3/H)x + G^3 = 0$

4. The sum $\sum_{r=2}^{\infty} \frac{1}{r^2 - 1}$ is equal to:
(A) 1 (B) 3/4 (C) 4/3 (D) none

5. If a, a_1, a_2, a_3,..., a_{2n}, b are in A.P. and a, g_1, g_2, g_3,.....g_{2n}, b are in G.P. and h is the harmonic mean of a and b, then $\frac{a_1 + a_{2n}}{g_1 g_{2n}} + \frac{a_2 + a_{2n-1}}{g_2 g_{2n-1}} + ... + \frac{a_n + a_{n+1}}{g_n g_{n+1}}$ is equal to
(A) $\frac{2n}{h}$ (B) 2nh (C) nh (D) $\frac{n}{h}$

6. One side of an equilateral triangle is 24 cm. The mid–points of its sides are joined to form another triangle whose mid – points are in turn joined to form still another triangle. This process continues indefinitely. Then the sum of the perimeters of all the triangles is
(A) 144 cm (B) 212 cm (C) 288 cm (D) none of these

7. If p is positive, then the sum to infinity of the series, $\frac{1}{1+p} - \frac{1-p}{(1+p)^2} + \frac{(1-p)^2}{(1+p)^3}$ -...... is:
(A) 1/2 (B) 3/4 (C) 1 (D) none of these

8. In a G.P. of positive terms, any term is equal to the sum of the next two terms. The common ratio of the G.P. is
(A) 2 cos 18° (B) sin 18° (C) cos 18° (D) 2 sin 18°

9. If $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...$ upto $\infty = \frac{\pi^2}{6}$, then $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... =$
(A) $\pi^2/12$ (B) $\pi^2/24$ (C) $\pi^2/8$ (D) none of these

10. The sum to 10 terms of the series $\sqrt{2} + \sqrt{6} + \sqrt{18} + \sqrt{54} + ...$ is
(A) $121(\sqrt{6} + \sqrt{2})$ (B) $\frac{121}{2}(\sqrt{3} + 1)$ (C) $243(\sqrt{3} + 1)$ (D) $243(\sqrt{3} - 1)$

11. If a_1, a_2,... a_n are in A.P. with common difference d ≠ 0, then the sum of the series
(sin d) [cosec a_1 cosec a_2 + cosec a_2 cosec a_3 + ... + cosec a_{n-1} cosec a_n]
(A) sec a_1 – sec a_n (B) cosec a_1 – cosec a_n (C) cot a_1 – cot a_n (D) tan a_1 – tan a_n

12. Sum of the series
$S = 1^2 - 2^2 + 3^2 - 4^2 + ... - 2002^2 + 2003^2$ is
(A) 2007006 (B) 1005004 (C) 2000506 (D) none of these

13. If $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ...+ \frac{1}{n}$, then value of $1 + \frac{3}{2} + \frac{5}{3} + ... + \frac{2n-1}{n}$ is
(A) $2n - H_n$ (B) $2n + H_n$ (C) $H_n - 2n$ (D) $H_n + n$

14. The sum of the series $\frac{1}{\log_2 4} + \frac{1}{\log_4 4} + \frac{1}{\log_8 4} + ... + \frac{1}{\log_{2^n} 4}$ is
(A) $\frac{1}{2}n(n + 1)$ (B) $\frac{1}{12}n(n + 1)(2n + 1)$ (C) $\frac{1}{n(n+1)}$ (D) $\frac{1}{4}n(n + 1)$

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15. If S_1, S_2, S_3 are the sums of first n natural numbers, their squares, their cubes respectively, then $\frac{S_3(1+8S_1)}{S_2^2}$ is equal to
(A) 1 (B) 3 (C) 9 (D) 10.

16. If p and q are respectively the sum and the sum of the squares of n successive integers beginning with 'a', then $nq - p^2$ is
(A) independent of 'a' (B) independent of 'n' (C) dependent on 'a' (D) none of these

17. Sum of n terms of the series $1 + \frac{x}{a_1} + \frac{x(x+a_1)}{a_1a_2} + \frac{x(x+a_1)(x+a_2)}{a_1a_2a_3} + \ldots$ is
(A) $\frac{x(x+a_1)\ldots(x+a_{n-1})}{a_1a_2\ldots a_3}$ (B) $\frac{(x+a_1)(x+a_2)\ldots(x+a_{n-1})}{a_1a_2\ldots a_{n-1}}$ (C) $\frac{x(x+a_1)\ldots(x+a_n)}{a_1a_2\ldots a_n}$ (D) none of these

18. $\{a_n\}$ and $\{b_n\}$ are two sequences given by $a_n = (x)^{1/2^n} + (y)^{1/2^n}$ and $b_n = (x)^{1/2^n} - (y)^{1/2^n}$ for all $n \in N$. The value of $a_1a_2a_3\ldots\ldots a_n$ is equal to
(A) $x - y$ (B) $\frac{x+y}{b_n}$ (C) $\frac{x-y}{b_n}$ (D) $\frac{xy}{b_n}$

19. If $a_1, a_2, a_3, \ldots\ldots, a_n$ are positive real numbers whose product is a fixed number c, then the minimum value of $a_1 + a_2 + a_3 + \ldots + a_{n-1} + 2a_n$ is **[IIT - 2002, 3]**
(A) $n(2c)^{1/n}$ (B) $(n+1)\,c^{1/n}$ (C) $2nc^{1/n}$ (D) $(n+1)(2c)^{1/n}$

Part : (B) May have more than one options correct

20. If $\sum_{r=1}^{n} r(r+1)(2r+3) = an^4 + bn^3 + cn^2 + dn + e$, then
(A) $a + c = b + d$ (B) $e = 0$ (C) a, $b - 2/3$, $c - 1$ are in A.P. (D) c/a is an integer

21. The sides of a right triangle form a G.P. The tangent of the smallest angle is
(A) $\sqrt{\frac{\sqrt{5}+1}{2}}$ (B) $\sqrt{\frac{\sqrt{5}-1}{2}}$ (C) $\sqrt{\frac{2}{\sqrt{5}+1}}$ (D) $\sqrt{\frac{2}{\sqrt{5}-1}}$

22. Sum to n terms of the series $S = 1^2 + 2(2)^2 + 3^2 + 2(4^2) + 5^2 + 2(6^2) + \ldots$ is
(A) $\frac{1}{2}n(n+1)^2$ when n is even (B) $\frac{1}{2}n^2(n+1)$ when n is odd
(C) $\frac{1}{4}n^2(n+2)$ when n is odd (D) $\frac{1}{4}n(n+2)^2$ when n is even.

23. If a, b, c are in H.P., then:
(A) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P. (B) $\frac{2}{b} = \frac{1}{b-a} + \frac{1}{b-c}$
(C) $a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$ are in G.P. (D) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

24. If b_1, b_2, b_3 ($b_i > 0$) are three successive terms of a G.P. with common ratio r, the value of r for which the inequality $b_3 > 4b_2 - 3b_1$ holds is given by
(A) $r > 3$ (B) $r < 1$ (C) $r = 3.5$ (D) $r = 5.2$

EXERCISE–5

1. If a, b, c are in A.P., then show that:
(i) $a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ are also in A.P. (ii) $b + c - a$, $c + a - b$, $a + b - c$ are in A.P.

2. If a, b, c, d are in G.P., prove that :
(i) $(a^2 - b^2)$, $(b^2 - c^2)$, $(c^2 - d^2)$ are in G.P. (ii) $\frac{1}{a^2+b^2}, \frac{1}{b^2+c^2}, \frac{1}{c^2+d^2}$ are in G.P.

3. Using the relation A.M. $\geq$ G.M. prove that
(i) $\tan\theta + \cot\theta \geq 2$; if $0 < \theta < \frac{\pi}{2}$ (ii) $(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) > 9x^2y^2z^2$.
(iii) $(a + b) . (b + c) . (c + a) \geq abc$; if a, b, c are positive real numbers

4. Find the sum in the n^{th} group of sequence,
(i) 1, (2, 3); (4, 5, 6, 7); (8, 9,........, 15); .... (ii) (1), (2, 3, 4), (5, 6, 7, 8, 9),........

5. If n is a root of the equation $x^2(1 - ac) - x(a^2 + c^2) - (1 + ac) = 0$ & if n HM's are inserted between a & c, show that the difference between the first & the last mean is equal to $ac(a - c)$.

6. The sum of the first ten terms of an AP is 155 & the sum of first two terms of a GP is 9. The first term of the AP is equal to the common ratio of the GP & the first term of the GP is equal to the common difference of the AP. Find the two progressions.

7. Find the sum of the series $\frac{5}{13} + \frac{55}{(13)^2} + \frac{555}{(13)^3} + \frac{5555}{(13)^4} + \ldots$ up to ∞

8. If $0 < x < \pi$ and the expression
$\exp\{(1 + |\cos x| + \cos^2 x + |\cos^3 x| + \cos^4 x + \ldots\ldots \text{ upto } \infty) \log_e 4\}$
satisfies the quadratic equation $y^2 - 20y + 64 = 0$ the find the value of x.

9. In a circle of radius R a square is inscribed, then a circle is inscribed in the square, a new square in the circle and so on for n times. Find the limit of the sum of areas of all the circles and the limit of the sum of areas of all the squares as $n \to \infty$.

10. The sum of the squares of three distinct real numbers, which are in GP is S^2. If their sum is αS, show that $\alpha^2 \in (1/3, 1) \cup (1, 3)$.

11. Let $S_1, S_2, \ldots S_p$ denote the sum of an infinite G.P. with the first terms 1, 2, ..., p and common ratios

1/2, 1/3, ..., 1/(p + 1) respectively. Show that $S_1 + S_2 + ... + S_p = \frac{1}{2}\,p(p+3)$

12. Circles are inscribed in the acute angle α so that every neighbouring circles touch each other. If the radius of the first circle is R then find the sum of the radii of the first n circles in terms of R and α.

13. Given that α, γ are roots of the equation, $A x^2 - 4x + 1 = 0$ and β, δ the roots of the equation, $B x^2 - 6x + 1 = 0$, find values of A and B, such that α, β, γ & δ are in H.P.

14. The airthmetic mean between m and n and the geometric mean between a and b are each equal to $\frac{ma+nb}{m+n}$: find the m and n in terms of a and b.

15. If a, b, c are positive real numbers then prove that (i) $b^2c^2 + c^2a^2 + a^2b^2 > abc\,(a + b + c)$.
(ii) $(a + b + c)^3 > 27abc$. (iii) $(a + b + c)^3 > 27\,(a + b - c)\,(c + a - b)\,(b + c - a)$

16. If 's' be the sum of 'n' positive unequal quantities a, b, c,......., then $\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}+... > \frac{n^2}{n-1}$.

17. Sum the following series to n terms and to infinity: (i) $\sum_{r=1}^{n} r\,(r+1)\,(r+2)\,(r+3)$

(ii) $\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+........$ (iii) $\frac{1}{3.5}+\frac{16}{3^2.5^2}+\frac{1}{5\,.\,7}+\frac{24}{5^2\,.\,7^2}+\frac{1}{7.9}+\frac{32}{7^2.9^2}+........$

18. Let a, b, c d be real numbers in G.P. If u, v, w, satisfy the system of equations
u + 2v + 3w = 6; 4u + 5v + 6w = 12
6u + 9v = 4 then show that the roots of the equation
$\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]\,x + u + v + w = 0$ and
$20x^2 + 10\,(a-d)^2\,x - 9 = 0$ are reciprocals of each other. **[IIT- 1999, 10]**

19. The fourth power of the common difference of an arithmetic progression with integer entries added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer. **[IIT - 2000, 4]**

20. If a, b & c are in arithmetic progression and a^2, b^2 & c^2 are in harmonic progression, then prove that either a = b = c or a, b & $-\frac{c}{2}$ are in geometric progression. **[IIT – 2003, 4]**

ANSWER KEY
EXERCISE–1

Q 1. 1
Q 3. $\mu = 14$
Q 4. $S = (7/81)\{10^{n+1} - 9n - 10\}$
Q 5. 35/222
Q 6. $n\,(n+1)/2\,(n^2 + n + 1)$
Q 7. 27
Q 10. $(14n - 6)/(8n + 23)$
Q 11. 1
Q 14. (a) 9 ; (b) 12
Q 16. a = 5 , b = 8 , c = 12
Q 18. (8 , –4 , 2 , 8)
Q 19. n^2
Q 20. **(i)** $2^{n+1} - 3$; $2^{n+2} - 4 - 3n$ **(ii)** $n^2 + 4n + 1$; $(1/6)\,n\,(n+1)\,(2n+13) + n$
Q 21. 120 , 30
Q 22. 6 , 3
Q 23. **(i)** $s_n = (1/24) - [1/\{6(3n+1)\,(3n+4)\}]$; $s_\infty = 1/24$ **(ii)** $(1/5)\,n\,(n+1)\,(n+2)\,(n+3)\,(n+4)$
(iii) $n/(2n+1)$ **(iv)** $S_n = 2\left[\frac{1}{2} - \frac{1.3.5.....(2n-1)(2n+1)}{2.4.6......(2n)(2n+2)}\right]$; $S_\infty = 1$
Q 24. **(a)** $(6/5)\,(6^n - 1)$ **(b)** $[n\,(n+1)\,(n+2)]/6$

EXERCISE–2

Q 6. 8 problems , 127.5 minutes
Q.8 C = 9 ; (3, –3/2 , –3/5)
Q 12. (iii) b = 4 , c = 6 , d = 9 OR b = – 2 , c = – 6 , d = – 18
Q.13 2499
Q 15. **(a)** a = 1 , b = 9 OR b = 1 , a = 9 ; **(b)** a = 1 ; b = 3 or vice versa
Q.19 $2p^3 - 9pq + 27r = 0$; roots are 1, 4, 7
Q 23. **(a)** $1 - \frac{x^n}{(x+1)\,(x+2)\,.....\,(x+n)}$ **(b)** $1 - \frac{1}{(1+a_1)\,(1+a_2)\,.....\,(1+a_n)}$
Q 24. n = 38
Q 25. 931

EXERCISE–3

Q 1. $\frac{1}{4}\,(2n-1)\,(n+1)^2$
Q 2. $S = \frac{1+ab}{(1-ab)^2}$ Where $a = 1 - x^{-1/3}$ & $b = 1 - y^{-1/4}$
Q3. $\beta \le (1/3)$; $\gamma \ge -(1/27)$
Q 4. – 3, 77
Q 5. 8, 24, 72, 216, 648
Q 6. **(a)** C **(b)** B
Q 7. **(a)** B **(b)** D

Q 8. $r = \pm 1/9$; $n = 2$; $a = 144/180$ **OR** $r = \pm 1/3$; $n = 4$; $a = 108$ **OR** $r = 1/81$; $n = 1$; $a = 160$

Q 9. **(a)** D **(b)** A

Q 10. A = 3 ; B = 8

Q 11. A.P.

Q 12. $x = 2\sqrt{2}$ and y = 3

Q 13. **(a)** A, **(b)** C, **(c)** D, **(d)** $\left[\left(A_1, A_2, \ldots\ldots\ldots\ldots A_n\right)\left(H_1, H_2, \ldots\ldots\ldots\ldots H_n\right)\right]^{\frac{1}{2n}}$

Q14. **(a)** D

Q.16 B

Q.18 **(a)** C, **(b)** n = 7

Q.19 $n_0 = 5$

EXERCISE–4

1. D **2.** A **3.** B **4.** B **5.** A **6.** A **7.** A **8.** D **9.** C **10.** A
11. C **12.** A **13.** A **14.** D **15.** C **16.** A **17.** B **18.** C **19.** A **20.** ABCD
21. BC **22.** AB **23.** ABCD **24.** ABCD

EXERCISE–5

4. (i) $2^{n-2}(2^n + 2^{n-1} - 1)$ (ii) $(n-1)^3 + n^3$

6. (3 + 6 + 12 +......); (2/3 + 25/3 + 625/6 +......)

7. $\frac{65}{36}$

8. $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{3}$

9. $2\pi R^2$; $4R^2$

12. $\frac{R\left(1-\sin\frac{\alpha}{2}\right)}{2\sin\frac{\alpha}{2}}\left[\left(\frac{1+\sin\frac{\alpha}{2}}{1-\sin\frac{\alpha}{2}}\right)^n - 1\right]$

13. A = 3; B = 8

14. $m = \frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}, n = \frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

17. (i) (1/5) n (n + 1) (n + 2) (n + 3) (n + 4)

(ii) $\frac{n(n+1)}{2(n^2+n+1)}$; $s_\infty = \frac{1}{2}$

(iii) $\frac{n}{3(2n+3)} + \frac{4}{9}\frac{n(n+3)}{(2n+3)^2}$